

# Existence and uniqueness of invariant measures for SPDEs with two reflecting walls <sup>\*</sup>

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## Abstract

In this article, we study stochastic partial differential equations with two reflecting walls  $h^1$  and  $h^2$ , driven by space-time white noise with non-constant diffusion coefficients under periodic boundary conditions. The existence and uniqueness of invariant measures is established under appropriate conditions. The strong Feller property is also obtained.

*Key Words:* stochastic partial differential equations with two reflecting walls; white noise; heat equation; invariant measures; coupling; strong Feller property.

*MSC:* Primary 60H15; Secondary 60J35

## 1 Introduction

Consider the following stochastic partial differential equations (SPDEs) with two reflecting walls

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t)) + \sigma(u(x,t))\dot{W}(x,t) \\ \quad + \eta(x,t) - \xi(x,t); \\ u(x,0) = u_0(x) \in C(S^1); \\ h^1(x) \leq u(x,t) \leq h^2(x), \text{ for } (x,t) \in Q. \end{cases} \quad (1.1)$$

$Q := S^1 \times \mathbb{R}_+$ ,  $S^1 := \mathbb{R}(\text{mod} 2\pi)$ , or  $\{e^{i\theta}; \theta \in \mathbb{R}\}$  denotes a circular ring and the random field  $W(x,t) := W(\{e^{i\theta}; 0 \leq \theta \leq x\} \times [0,t])$  is a regular Brownian sheet defined on a filtered probability space  $(\Omega, P, \mathcal{F}; \mathcal{F}_t)$ . The

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random measures  $\xi$  and  $\eta$  are added to equation (1.1) to prevent the solution from leaving the interval  $[h^1, h^2]$ .

We assume that the reflecting walls  $h^1(x)$ ,  $h^2(x)$  are continuous functions satisfying

(H1)  $h^1(x) < h^2(x)$  for  $x \in S^1$ ;

(H2)  $\frac{\partial^2 h^i}{\partial x^2} \in L^2(S^1)$ , where  $\frac{\partial^2}{\partial x^2}$  is interpreted in a distributional sense.

We also assume that the coefficients:  $f, \sigma : \mathbb{R} \rightarrow \mathbb{R}$  satisfy

(F1) there exists  $L > 0$  such that

$$|f(z_1) - f(z_2)| + |\sigma(z_1) - \sigma(z_2)| \leq L|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{R};$$

The following is the definition of a solution of a SPDE with two reflecting walls  $h^1, h^2$ .

**Definition 1.1.** A triplet  $(u, \eta, \xi)$  is a solution to the SPDE (1.1) if

(i)  $u = \{u(x, t); (x, t) \in Q\}$  is a continuous, adapted random field (i.e.,  $u(x, t)$  is  $\mathcal{F}_t$ -measurable  $\forall t \geq 0, x \in S^1$ ) satisfying  $h^1(x) \leq u(x, t) \leq h^2(x)$ , a.s;

(ii)  $\eta(dx, dt)$  and  $\xi(dx, dt)$  are positive and adapted (i.e.  $\eta(B)$  and  $\xi(B)$  is  $\mathcal{F}_t$ -measurable if  $B \subset S^1 \times [0, t]$ ) random measures on  $Q$  satisfying

$$\eta(S^1 \times [0, T]) < \infty, \quad \xi(S^1 \times [0, T]) < \infty$$

for  $T > 0$ ;

(iii) for all  $t \geq 0$  and  $\phi \in C^\infty(S^1)$  we have

$$\begin{aligned} & (u(t), \phi) - \int_0^t (u(s), \phi'') ds - \int_0^t (f(u(s)), \phi) ds - \int_0^t \int_{S^1} \phi(x) \sigma(u(x, s)) W(dx, ds) \\ &= (u_0, \phi(x)) + \int_0^t \int_{S^1} \phi(x) \eta(dx, ds) - \int_0^t \int_{S^1} \phi(x) \xi(dx, ds), \quad a.s, \end{aligned} \quad (1.2)$$

where  $(\cdot, \cdot)$  denotes the inner product in  $L^2(S^1)$  and  $u(t)$  denotes  $u(\cdot, t)$ ;

(iv)

$$\int_Q (u(x, t) - h^1(x)) \eta(dx, dt) = \int_Q (h^2(x) - u(x, t)) \xi(dx, dt) = 0.$$

The existence and uniqueness of the solution of equation (1.1) is established in [13], see also [11] for SPDEs with one reflecting barrier. SPDEs with reflection were first studied by Nualart and Pardoux in [4]. Interesting properties were obtained in [12].

The aim of this paper is to establish the existence and uniqueness of invariant measures, as well as the strong Feller property of fully non-linear SPDEs with two reflecting walls (1.1).

For SPDEs without reflection, the existence and uniqueness of invariant measures has been studied by many people, see Sowers [9], Mueller [3], Peszat and Zabczyk [7], Da Prato and Zabczyk [2]. For SPDEs with reflection, when the diffusion coefficient  $\sigma$  is a constant, existence and uniqueness of invariant measures was obtained by Otobe [5], [6]. The strong Feller property of SPDEs has been studied by several authors, see Peszat and Zabczyk [7], Da Prato and Zabczyk [2]. The strong Feller property of SPDEs with reflection at 0 was first proved in [14].

For the existence of invariant measures, our approach is to use Krylov-Bogolyubov theorem. To this end, the continuity of the solution with respect to the solutions of some random obstacle problems plays an important role. For the uniqueness, we adapted a coupling method used by Mueller [3]. Because of the reflection, we need to establish a kind of uniform coupling for approximating solutions. The strong Feller property of SPDEs with two reflecting walls will be obtained in a similar way as that that in Zhang [14].

The rest of the paper is organized as follows. In Section 2, we give the proof of the existence and uniqueness of invariant measures. Section 3 establishes the strong Feller property.

## 2 Existence and Uniqueness of Invariant Measures

Denote by  $\mathcal{B}(C(S^1))$  the  $\sigma$ -field of all Borel subsets of  $C(S^1)$  and by  $\mathcal{M}(C(S^1))$  the set of all probability measures defined on  $(C(S^1), \mathcal{B}(C(S^1)))$ . We denote by  $u(x, t, u_0)$  the solution of equation (1.1) and by  $P_t(u_0, \cdot)$  the corresponding transition function

$$P_t(u_0, \Gamma) = P(u(\cdot, t, u_0) \in \Gamma), \quad \Gamma \in \mathcal{B}(C(S^1)), \quad t > 0,$$

where  $u_0$  is the initial condition. For  $\mu \in \mathcal{M}(C(S^1))$  we set

$$P_t^* \mu(\Gamma) = \int_{C(S^1)} P_t(x, \Gamma) \mu(dx),$$

where  $t \geq 0$ ,  $\Gamma \in \mathcal{B}(C(S^1))$ .

**Definition 2.1.** *A probability measure  $\mu \in \mathcal{M}(C(S^1))$  is said to be invariant or stationary with respect to  $P_t$ ,  $t \geq 0$ , if and only if  $P_t^* \mu = \mu$  for each  $t \geq 0$ .*

The initial condition  $u_0(x)$  satisfies  
(F2)  $u_0(x) \in C(S^1)$  satisfy  $h^1(x) \leq u_0(x) \leq h^2(x)$ , for  $x \in S^1$ .

**Theorem 2.1.** *Suppose the hypotheses (H1)-(H2), (F1)-(F2) hold. Then there exists an invariant measure to equation (1.1) on  $C(S^1)$ .*

**Proof.** According to Krylov-Bogolyubov theorem (see [2]), if the family  $\{P_t(u_0, \cdot); t \geq 1\}$  is tight, then there exists an invariant measure for equation (1.1). So we need to show that for any  $\varepsilon > 0$  there is a compact set  $K \subset C(S^1)$  such that

$$P(u(t) \in K) \geq 1 - \varepsilon, \text{ for any } t \geq 1.$$

where  $u(t) = u(t, u_0) = u(\cdot, t, u_0)$ . On the other hand, for any  $t \geq 1$ , we have by the Markov property

$$P(u(t) \in K) = \mathbb{E}(P_1(u(t-1), K)). \quad (2.1)$$

Thus it is enough to show  $P(u(1, u(t-1)) \in K) \geq 1 - \varepsilon$ , for any  $t \geq 1$ . As  $h^1(\cdot) \leq u(t-1)(\cdot) \leq h^2(\cdot)$ , it suffices to find a compact subset  $K \subset C(S^1)$  such that

$$P_1(g, K) \geq 1 - \varepsilon, \text{ for all } g \in C(S^1) \text{ with } h^1 \leq g \leq h^2. \quad (2.2)$$

Put

$$\begin{aligned} v(x, t, g) &= \int_0^t \int_{S^1} G_{t-s}(x, y) f(u(y, s, g)) dy ds \\ &\quad + \int_0^t \int_{S^1} G_{t-s}(x, y) \sigma(u(y, s, g)) W(dy, ds), \end{aligned} \quad (2.3)$$

where  $G_t(x, y)$  is the Green's function of the heat equation on  $S^1$ . Then  $u$  can be written in the form(see [4], [1] and [10])

$$\begin{aligned} u(x, t, g) - \int_{S^1} G_t(x, y) g(y) dy &= v(x, t, g) + \int_0^t \int_{S^1} G_{t-s}(x, y) \eta(g)(dx, dt) \\ &\quad - \int_0^t \int_{S^1} G_{t-s}(x, y) \xi(g)(dx, dt), \end{aligned}$$

where  $\eta(g)$ ,  $\xi(g)$  indicates the dependence of the random measures on the initial condition  $g$ . Put

$$\bar{u}(x, t, g) = u(x, t, g) - \int_{S^1} G_t(x, y) g(y) dy$$

Then  $(\bar{u}, \eta, \xi)$  solves a random obstacle problem. From the relationship between  $\bar{u}$  and  $v$  proved in Theorem 4.1 in [13], we have the following inequality

$$\|\bar{u}(g) - \bar{u}(\hat{g})\|_\infty^1 \leq 2\|v(g) - v(\hat{g})\|_\infty^1,$$

where  $\|\omega\|_\infty^1 := \sup_{x \in S^1, t \in [0,1]} |\omega(x, t)|$ . So  $\bar{u}$  is a continuous functional of  $v$  and

denoted by  $u = \Phi(v)$ , where  $\Phi(\cdot) : C(S^1 \times [0, 1]) \rightarrow C(S^1 \times [0, 1])$  is continuous. In particular,  $\bar{u}(\cdot, 1, g)$  is also a continuous functional of  $v$ , from  $C(S^1 \times [0, 1])$  to  $C(S^1)$ . We denote this functional by  $\Phi_1$ , i.e.  $\bar{u}(\cdot, 1, g) = \Phi_1(v(\cdot, g))$ , where  $v(\cdot, g) = v(\cdot, \cdot, g)$ . If  $K''$  is a compact subset of  $C(S^1 \times [0, 1])$ , then  $K' = \Phi_1(K'')$  is a compact subset in  $C(S^1)$  and

$$\begin{aligned} P(\bar{u}(\cdot, 1, g) \in K') &= P(\bar{u}(\cdot, 1, g) \in \Phi_1(K'')) \\ &\geq P(v(\cdot, g) \in K''). \end{aligned} \quad (2.4)$$

Next, we want to find a compact set  $K''(\subset C(S^1 \times [0, 1]))$  such that

$$P(v(\cdot, g) \in K'') \geq 1 - \varepsilon, \text{ for all } g \in C(S^1) \text{ with } h^1 \leq g \leq h^2. \quad (2.5)$$

For  $0 < \alpha < \frac{1}{4}$  and  $\kappa > 0$ , from Proposition A.1 in [8] and using a similar proof to that of Corollary 3.4 in [10], there exists a random variable  $Y(g)$  such that with probability one, for all  $x, y \in S^1$  and  $s, t \in (0, 1]$ ,

$$|v(x, t, g) - v(y, s, g)| \leq Y(g)(d((x, t), (y, s)))^{\alpha - \kappa} \text{ and } \mathbb{E}(Y(g))^{\frac{1}{\kappa}} \leq C_0, \quad (2.6)$$

where  $d((x, t), (y, s)) := (r^2(x, y) + (t - s)^2)^{\frac{1}{2}}$  with  $r(x, y)$  the length of the shortest arc of  $S^1$  connecting  $x$  with  $y$  and  $C_0$  is independent of  $g$ .

Define

$$\begin{aligned} \|v\|_\alpha &= \sup \left\{ \frac{|v(x, t) - v(y, s)|}{d^\alpha((x, t), (y, s))}; \right. \\ &\quad \left. (x, t), (y, s) \in S^1 \times [0, 1], (x, t) \neq (y, s) \right\}, \text{ for } \alpha < \frac{1}{4}. \end{aligned}$$

By the Arzela-Ascoli theorem, for all  $r > 0$ ,  $K_r := \{v; \|v\|_\alpha \leq r\}$  is a compact subset of  $C(S^1 \times [0, 1])$ . In view of (2.6), we see that for given  $\varepsilon > 0$ , there exists  $r_0$  such that

$$P(v(\cdot, g) \in K_{r_0}^c) \leq \varepsilon, \text{ for all } g \text{ with } h^1 \leq g \leq h^2.$$

Choosing  $K'' = K_{r_0}$ , we obtain (2.5). Hence  $P(\bar{u}(\cdot, 1, g) \in K') \geq 1 - \varepsilon$  for all  $g \in C(S^1)$  with  $h^1 \leq g \leq h^2$ . On the other hand, it is easy to see that there is a compact subset  $K_0 \subset C(S^1)$  such that

$$\left\{ \int_{S^1} G_1(x, y)g(y)dy; \quad h^1 \leq g \leq h^2 \right\} \subset K_0$$

Define  $K = K' + K_0$ . We have

$$P_1(g, K) = P(u(\cdot, 1, g) \in K) \geq P(\bar{u}(\cdot, 1, g) \in K') \geq 1 - \varepsilon,$$

for all  $g \in C(S^1)$  with  $h^1 \leq g \leq h^2$ . This finishes the proof.  $\square$

For the uniqueness of invariant measures, we need the following proposition. For simplicity, we put  $u(x, t) = u(x, t, u_0)$ .

**Proposition 2.1.** *Under the assumption in Theorem 2.1, for any  $p \geq 1$ ,  $T > 0$ ,  $\sup_{\varepsilon, \delta} \mathbb{E}(\|u^{\varepsilon, \delta}\|_\infty^T)^p < \infty$  and  $u^{\varepsilon, \delta}$  converges uniformly on  $S^1 \times [0, T]$  to  $u$  as  $\varepsilon, \delta \rightarrow 0$  a.s, where  $u$ ,  $u^{\varepsilon, \delta}$  are the solutions of equation (1.1) and the penalized SPDEs*

$$\begin{cases} \frac{\partial u^{\varepsilon, \delta}(x, t)}{\partial t} = \frac{\partial^2 u^{\varepsilon, \delta}(x, t)}{\partial x^2} + f(u^{\varepsilon, \delta}(x, t)) + \sigma(u^{\varepsilon, \delta}(x, t))\dot{W}(x, t) \\ \quad + \frac{1}{\delta}(u^{\varepsilon, \delta}(x, t) - h^1(x))^- - \frac{1}{\varepsilon}(u^{\varepsilon, \delta}(x, t) - h^2(x))^+; \\ u^{\varepsilon, \delta}(x, 0) = u_0(x). \end{cases}$$

**Proof.** Let  $v^{\varepsilon, \delta}$  be the solution of equation

$$\begin{cases} \frac{\partial v^{\varepsilon, \delta}(x, t)}{\partial t} = \frac{\partial^2 v^{\varepsilon, \delta}(x, t)}{\partial x^2} + f(u^{\varepsilon, \delta}(x, t)) + \sigma(u^{\varepsilon, \delta}(x, t))\dot{W}(x, t); \\ v^{\varepsilon, \delta}(x, 0) = u_0(x). \end{cases} \quad (2.7)$$

Set  $\bar{\Phi}^{\varepsilon, \delta}(t) = \sup_{s \leq t, y \in S^1} (v^{\varepsilon, \delta}(y, s) - h^2(y))^+$ . Note that  $\bar{\Phi}^{\varepsilon, \delta}(t)$  is increasing w.r.t.  $t$  and  $v^{\varepsilon, \delta} - \bar{\Phi}^{\varepsilon, \delta} \leq h^2$ .  $\bar{z}^{\varepsilon, \delta}(x, t) := v^{\varepsilon, \delta}(x, t) - \bar{\Phi}^{\varepsilon, \delta}(t) - u^{\varepsilon, \delta}(x, t)$  is a solution of equation

$$\begin{cases} \frac{\partial \bar{z}^{\varepsilon, \delta}}{\partial t} + \frac{\partial \bar{\Phi}^{\varepsilon, \delta}}{\partial t} = \frac{\partial^2 \bar{z}^{\varepsilon, \delta}}{\partial x^2} - \frac{1}{\delta}(u^{\varepsilon, \delta} - h^1)^- + \frac{1}{\varepsilon}(u^{\varepsilon, \delta} - h^2)^+; \\ \bar{z}^{\varepsilon, \delta}(x, 0) = 0. \end{cases} \quad (2.8)$$

Multiplying (2.8) by  $(\bar{z}^{\varepsilon, \delta})^+$  and using  $((u^{\varepsilon, \delta} - h^2)^+, (\bar{z}^{\varepsilon, \delta})^+) = 0$  we get  $(\bar{z}^{\varepsilon, \delta})^+ = 0$ . Hence,

$$u^{\varepsilon, \delta} \geq v^{\varepsilon, \delta} - \bar{\Phi}^{\varepsilon, \delta}.$$

Similarly, setting  $\bar{z}^{\varepsilon, \delta}(x, t) = u^{\varepsilon, \delta}(x, t) - v^{\varepsilon, \delta}(x, t) - \sup_{s \leq t, y \in S^1} (v^{\varepsilon, \delta}(y, s) - h^1(s))^-$ , we can show that

$$u^{\varepsilon, \delta} \leq v^{\varepsilon, \delta} + \sup_{s \leq t, y \in S^1} (v^{\varepsilon, \delta} - h^1)^-.$$

As  $\sup_{\varepsilon, \delta} \mathbb{E}(\|v^{\varepsilon, \delta}\|_{\infty}^T)^p < \infty$ , the above two inequalities implies

$$\sup_{\varepsilon, \delta} \mathbb{E}(\|u^{\varepsilon, \delta}\|_{\infty}^T)^p < \infty.$$

Since  $u^{\varepsilon, \delta}$  is increasing in  $\delta$  by the comparison theorem of SPDEs (see [1]), we can show  $u^{\varepsilon} := \lim_{\delta \downarrow 0} u^{\varepsilon, \delta}$  exists a.s. and  $u^{\varepsilon}$  solves

$$\begin{cases} \frac{\partial u^{\varepsilon}(x, t)}{\partial t} = \frac{\partial^2 u^{\varepsilon}(x, t)}{\partial x^2} + f(u^{\varepsilon}(x, t)) + \sigma(u^{\varepsilon}(x, t))\dot{W}(x, t) \\ \quad + \eta^{\varepsilon}(x, t) - \frac{1}{\varepsilon}(u^{\varepsilon}(x, t) - h^2(x))^+; \\ u^{\varepsilon}(x, 0) \geq h^1(x); \\ u^{\varepsilon}(x, 0) = u_0(x), \end{cases} \quad (2.9)$$

where  $\eta^{\varepsilon}(dx, dt) := \lim_{\delta \downarrow 0} \frac{(u^{\varepsilon, \delta}(x, t) - h^1(x))^+}{\delta} dx dt$ . Also, by comparison, we know that  $u^{\varepsilon}$  is decreasing as  $\varepsilon \downarrow 0$ . Let  $v^{\varepsilon}$  be the solution of equation (2.7) replacing  $u^{\varepsilon, \delta}$  by  $u^{\varepsilon}$ . Setting  $\bar{z}^{\varepsilon}(x, t) = u^{\varepsilon}(x, t) - v^{\varepsilon}(x, t) - \sup_{s \leq t, y \in S^1} (v^{\varepsilon}(y, s) - h^1(y))^-$ , we can show

$$u^{\varepsilon} \leq v^{\varepsilon} + \sup_{s \leq t, y \in S^1} (v^{\varepsilon} - h^1)^-.$$

In addition, by the definition of  $u^{\varepsilon}$ ,  $u^{\varepsilon} \geq h^1$ . Hence,  $u := \lim_{\varepsilon \downarrow 0} u^{\varepsilon} = \lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} u^{\varepsilon, \delta}$  exists a.s.

The continuity of  $u$  can be proved similarly as in Theorem 4.1 in [1]. The uniform convergence of  $u^{\varepsilon, \delta}$  w.r.t.  $(x, t)$  follows from Dini's theorem.  $\square$

The following result is the uniqueness of invariant measures.

**Theorem 2.2.** *Under the assumptions in Theorem 2.1 and that  $\sigma \geq L_0$  for some constant  $L_0 > 0$ , there is a unique invariant measure for the equation (1.1).*

**Proof.** We will adopt the coupling method used in Mueller [3] to SPDEs with reflection. Let  $u^1(x, 0)$  and  $u^2(x, 0)$  be two initial values having distributions given by two invariant probabilities  $\mu_1$  and  $\mu_2$ . Then  $u^1(x, t)$  and  $u^2(x, t)$  also have these distributions for any  $t > 0$ . Thus

$$\text{Var}(\mu_1 - \mu_2) \leq P\left(\sup_{x \in S^1} |u^1(x, t) - u^2(x, t)| \neq 0\right).$$

Thus, for given two initial functions  $u^1(x, 0)$  and  $u^2(x, 0)$ , it is sufficient to construct two coupled processes  $u^1(x, t)$ ,  $u^2(x, t)$  satisfying equation (1.1), driven by different white noises on a probability space  $(\Omega, \mathcal{F}, P)$ , such that

$$\lim_{t \rightarrow \infty} P\left(\sup_{x \in S^1} |u^1(x, t) - u^2(x, t)| \neq 0\right) = 0. \quad (2.10)$$

We first assume  $u^1(x, 0) \geq u^2(x, 0)$ ,  $x \in S^1$ . We want to construct two independent space-time white noises  $W_1(x, t)$ ,  $W_2(x, t)$  defined on a probability space  $(\Omega, \mathcal{F}, P)$ , and a solution  $u, v$  of the following SPDEs with two reflecting walls

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t)) + \sigma(u(x, t)) \dot{W}_1(x, t) \\ &\quad + \eta_1(x, t) - \xi_1(x, t), \\ \frac{\partial v(x, t)}{\partial t} &= \frac{\partial^2 v(x, t)}{\partial x^2} + f(v(x, t)) + \eta_2(x, t) - \xi_2(x, t) \\ &\quad + \sigma(v(x, t)) \left[ (1 - |u - v| \wedge 1)^{\frac{1}{2}} \dot{W}_1(x, t) + (|u - v| \wedge 1)^{\frac{1}{2}} \dot{W}_2(x, t) \right], \\ u(x, 0) &= u^1(x, 0), \quad v(x, 0) = u^2(x, 0). \end{aligned} \quad (2.11)$$

Note that the coefficients in the second equation in (2.11) is not Lipschitz. The existence of a solution of equation (2.11) is not automatic. In the following, using a similar method as that in the paper [3], we will give a construction of a solution on some probability space. The construction will also be used to prove the successful coupling

$$\lim_{t \rightarrow \infty} P\left(\sup_{x \in S^1} |u(x, t) - v(x, t)| \neq 0\right) = 0.$$

For  $0 \leq z \leq 1$ , set

$$\begin{aligned} f_n(z) &= \left(z + \frac{1}{n}\right)^{\frac{1}{2}} - \left(\frac{1}{n}\right)^{\frac{1}{2}}, \\ g_n(z) &= \left(1 - f_n(z)^2\right)^{\frac{1}{2}}. \end{aligned}$$

We have  $f_n(z)^2 + g_n(z)^2 = 1$  and that  $f_n(z) \rightarrow z^{\frac{1}{2}}$ ,  $g_n(z) \rightarrow (1-z)^{\frac{1}{2}}$  uniformly as  $n \rightarrow \infty$ , for  $z \in S^1$ .

Let  $\dot{\bar{W}}_1(x, t)$ ,  $\dot{\bar{W}}_2(x, t)$  be two independent space-time white noises defined on a probability space  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ . Let  $\bar{u}, \bar{v}^n$  be the unique solution of



the following SPDEs with two reflecting walls

$$\begin{aligned}
\frac{\partial \bar{u}(x, t)}{\partial t} &= \frac{\partial^2 \bar{u}(x, t)}{\partial x^2} + f(\bar{u}(x, t)) + \sigma(\bar{u}(x, t)) \dot{\bar{W}}_1(x, t) \\
&\quad + \bar{\eta}_1(x, t) - \bar{\xi}_1(x, t), \\
\frac{\partial \bar{v}^n(x, t)}{\partial t} &= \frac{\partial^2 \bar{v}^n(x, t)}{\partial x^2} + f(\bar{v}^n(x, t)) + \bar{\eta}_2^n(x, t) - \bar{\xi}_2^n(x, t) \\
&\quad + \sigma(\bar{v}^n(x, t)) [g_n(|\bar{u} - \bar{v}^n| \wedge 1) \dot{\bar{W}}_1(x, t) + f_n(|\bar{u} - \bar{v}^n| \wedge 1) \dot{\bar{W}}_2(x, t)], \\
\bar{u}(x, 0) &= u^1(x, 0), \quad \bar{v}(x, 0) = u^2(x, 0).
\end{aligned} \tag{2.12}$$

The existence and uniqueness of  $(\bar{u}, \bar{v}^n)$  is guaranteed because of the Lipschitz continuity of the coefficients. Put

$$\begin{aligned}
\hat{u}(x, t) &= \int_{S^1} G_t(x, y) u^1(y, 0) dy + \int_0^t \int_{S^1} G_{t-s}(x, y) f(\bar{u}(y, s)) dy ds \\
&\quad + \int_0^t \int_{S^1} G_{t-s}(x, y) \sigma(\bar{u}(y, s)) \dot{\bar{W}}_1(dy, ds)
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
\hat{v}^n(x, t) &= \int_{S^1} G_t(x, y) u^2(y, 0) dy + \int_0^t \int_{S^1} G_{t-s}(x, y) f(\bar{v}^n(y, s)) dy ds \\
&\quad + \int_0^t \int_{S^1} G_{t-s}(x, y) \sigma(\bar{v}^n(y, s)) \dot{\bar{W}}_2(dy, ds),
\end{aligned} \tag{2.14}$$

where

$$\dot{W}^n(x, t) = [g_n(|\bar{u} - \bar{v}^n| \wedge 1) \dot{\bar{W}}_1(x, t) + f_n(|\bar{u} - \bar{v}^n| \wedge 1) \dot{\bar{W}}_2(x, t)]$$

is another space-time white noise on  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ . From the proof of Theorem 2.1, it is known that there exists a continuous functional  $\Phi$  from  $C(S^1 \times [0, T])$  into  $C(S^1 \times [0, T])$  (for any  $T > 0$ ) such that  $\bar{u} = \Phi(\hat{u})$  and  $\bar{v}^n = \Phi(\hat{v}^n)$ . On the other hand, following the same proof of Lemma 3.1 in [3] it can be shown that the sequence  $\hat{u}, \hat{v}^n, n \geq 1$  is tight. As the images under the continuous map  $\Phi$ , the vector  $(\bar{u}, \bar{v}^n, \bar{W}_1, \bar{W}_2)$  is also tight. By Skorohod's representation theorem, there exist random fields  $(u, v^n, W_1, W_2)$ ,  $n \geq 1$  on some probability space  $(\Omega, \mathcal{F}, P)$  such that  $(u, v^n, W_1, W_2)$  has the same law as  $(\bar{u}, \bar{v}^n, \bar{W}_1, \bar{W}_2)$  and that the following SPDEs with two reflecting walls

hold

$$\begin{aligned}
\frac{\partial u(x, t)}{\partial t} &= \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t)) + \sigma(u(x, t))\dot{W}_1(x, t) \\
&\quad + \eta_1(x, t) - \xi_1(x, t), \\
\frac{\partial v^n(x, t)}{\partial t} &= \frac{\partial^2 v^n(x, t)}{\partial x^2} + f(v^n(x, t)) + \eta_2^n(x, t) - \xi_2^n(x, t) \\
&\quad + \sigma(v^n(x, t)) [g_n(|u - v^n| \wedge 1)\dot{W}_1(x, t) + f_n(|u - v^n| \wedge 1)\dot{W}_2(x, t)], \\
u(x, 0) &= u^1(x, 0), \quad v^n(x, 0) = u^2(x, 0).
\end{aligned} \tag{2.15}$$

Furthermore,  $v^n \rightarrow v$  uniformly almost surely as  $n \rightarrow \infty$ . By a similar proof as that of Theorem 4.1 in [13] we can prove that the limit  $(u, v)$  satisfies the following SPDEs with two reflecting walls

$$\begin{aligned}
\frac{\partial u(x, t)}{\partial t} &= \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t)) + \sigma(u(x, t))\dot{W}_1(x, t) \\
&\quad + \eta_1(x, t) - \xi_1(x, t), \\
\frac{\partial v(x, t)}{\partial t} &= \frac{\partial^2 v(x, t)}{\partial x^2} + f(v(x, t)) + \eta_2(x, t) - \xi_2(x, t) \\
&\quad + \sigma(v(x, t)) [(1 - |u - v| \wedge 1)^{\frac{1}{2}}\dot{W}_1(x, t) + (|u - v| \wedge 1)^{\frac{1}{2}}\dot{W}_2(x, t)], \\
u(x, 0) &= u^1(x, 0), \quad v(x, 0) = u^2(x, 0).
\end{aligned} \tag{2.16}$$

The next step is to show that  $u, v$  admits a successful coupling. To this end, consider the following approximating SPDEs

$$\left\{ \begin{aligned} \frac{\partial u^{\varepsilon, \delta}}{\partial t} &= \frac{\partial^2 u^{\varepsilon, \delta}}{\partial x^2} + f(u^{\varepsilon, \delta}) + \frac{1}{\delta}(u^{\varepsilon, \delta} - h^1)^- - \frac{1}{\varepsilon}(u^{\varepsilon, \delta} - h^2)^+ + \sigma(u^{\varepsilon, \delta})\dot{W}_1; \\ \frac{\partial v^{n, \varepsilon, \delta}}{\partial t} &= \frac{\partial^2 v^{n, \varepsilon, \delta}}{\partial x^2} + f(v^{n, \varepsilon, \delta}) + \frac{1}{\delta}(v^{n, \varepsilon, \delta} - h^1)^- - \frac{1}{\varepsilon}(v^{n, \varepsilon, \delta} - h^2)^+ \\ &\quad + \sigma(v^{n, \varepsilon, \delta}) [g_n(|u^{\varepsilon, \delta} - v^{n, \varepsilon, \delta}| \wedge 1)\dot{W}_1(x, t) \\ &\quad + f_n(|u^{\varepsilon, \delta} - v^{n, \varepsilon, \delta}| \wedge 1)\dot{W}_2(x, t)]; \\ u^{\varepsilon, \delta}(x, 0) &= u^1(x, 0), \quad v^{n, \varepsilon, \delta}(x, 0) = u^2(x, 0). \end{aligned} \right. \tag{2.17}$$

We may and will assume that  $f(u)$  is non-increasing. Otherwise, we consider  $\tilde{u} := e^{-Lt}u$ ,  $\tilde{v} := e^{-Lt}v$ , where  $L$  is the Lipschitz constant in (F1), which

satisfy

$$\begin{aligned}
\frac{\partial \tilde{u}(x, t)}{\partial t} &= \frac{\partial^2 \tilde{u}(x, t)}{\partial x^2} + e^{-Lt} f(e^{Lt} \tilde{u}(x, t)) - L\tilde{u}(x, t) \\
&\quad + e^{-Lt} \sigma(e^{Lt} \tilde{u}(x, t)) \dot{W}_1(x, t) + \eta_3(x, t) - \xi_3(x, t), \\
\frac{\partial \tilde{v}^n(x, t)}{\partial t} &= \frac{\partial^2 \tilde{v}^n(x, t)}{\partial x^2} + e^{-Lt} f(e^{Lt} \tilde{v}^n(x, t)) - L\tilde{v}^n(x, t) + \eta_4^n(x, t) - \xi_4^n(x, t) \\
&\quad + e^{-Lt} \sigma(e^{Lt} \tilde{v}^n(x, t)) [g_n(|e^{Lt} \tilde{u} - e^{Lt} \tilde{v}^n| \wedge 1) \dot{W}_1(x, t) \\
&\quad + f_n(|e^{Lt} \tilde{u} - e^{Lt} \tilde{v}^n| \wedge 1) \dot{W}_2(x, t)], \\
u(x, 0) &= u^1(x, 0), \quad v^n(x, 0) = u^2(x, 0).
\end{aligned}$$

The new drift  $e^{-Lt} f(e^{Lt} x) - Lx$  is non-increasing. Also, if  $\tilde{u}, \tilde{v}$  satisfy a successful coupling, so does  $u, v$ . Note that all the coefficients in (2.17) are Lipschitz continuous. We can apply Proposition 2.1 to conclude that  $u^{\varepsilon, \delta}(x, t) \rightarrow u(x, t)$ ,  $v^{n, \varepsilon, \delta}(x, t) \rightarrow v^n(x, t)$  uniformly on  $S^1 \times [0, T]$  ( for any  $T > 0$ ) as  $\varepsilon, \delta \rightarrow 0$ . As  $u^1(x, 0) \geq u^2(x, 0)$ , as lemma 3.1 in [3], we can show that  $u^{\varepsilon, \delta} \geq v^{n, \varepsilon, \delta}$ . Let

$$U^{n, \varepsilon, \delta}(t) = \int_{S^1} (u^{\varepsilon, \delta}(x, t) - v^{n, \varepsilon, \delta}(x, t)) dx. \quad (2.18)$$

It follows from the above equation that

$$U^{n, \varepsilon, \delta}(t) = \int_{S^1} (u_1(x, 0) - u_2(x, 0)) dx + \int_0^t C^{n, \varepsilon, \delta}(s) ds + M^{n, \varepsilon, \delta}(t), \quad (2.19)$$

where

$$\begin{aligned}
C^{n, \varepsilon, \delta}(t) &= \int_{S^1} \left\{ f(u^{\varepsilon, \delta}) - f(v^{n, \varepsilon, \delta}) + \frac{1}{\delta} (u^{\varepsilon, \delta} - h^1)^-(x, t) - \frac{1}{\delta} (v^{n, \varepsilon, \delta} - h^1)^-(x, t) \right. \\
&\quad \left. - \left( \frac{1}{\varepsilon} (u^{\varepsilon, \delta} - h^2)^+(x, t) - \frac{1}{\varepsilon} (v^{n, \varepsilon, \delta} - h^2)^+(x, t) \right) \right\} dx \\
&\leq 0,
\end{aligned}$$

$$\begin{aligned}
M^{n, \varepsilon, \delta}(t) &= \int_0^t \int_{S^1} \sigma(u^{\varepsilon, \delta}(x, s)) W_1(dx, ds) \\
&\quad - \int_0^t \int_{S^1} \sigma(v^{n, \varepsilon, \delta}(x, s)) g_n(|u^{\varepsilon, \delta} - v^{n, \varepsilon, \delta}| \wedge 1) \dot{W}_1(dx, ds) \\
&\quad - \int_0^t \int_{S^1} \sigma(v^{n, \varepsilon, \delta}(x, s)) f_n(|u^{\varepsilon, \delta} - v^{n, \varepsilon, \delta}| \wedge 1) \dot{W}_2(dx, ds).
\end{aligned}$$

Observe that

$$\begin{aligned} & \lim_{\varepsilon, \delta \rightarrow 0} U^{n, \varepsilon, \delta}(t) \\ &= U^n(t) := \int_{S^1} (u(x, t) - v^n(x, t)) dx, \end{aligned} \quad (2.20)$$

and

$$\begin{aligned} & \lim_{\varepsilon, \delta \rightarrow 0} M^{n, \varepsilon, \delta}(t) \\ &= M^n(t) := \int_0^t \int_{S^1} \sigma(u(x, s)) W_1(dx, ds) \\ & \quad - \int_0^t \int_{S^1} \sigma(v^n(x, s)) g_n(|u - v^n| \wedge 1) \dot{W}_1(dx, ds) \\ & \quad - \int_0^t \int_{S^1} \sigma(v^n(x, s)) f_n(|u - v^n| \wedge 1) \dot{W}_2(dx, ds). \end{aligned} \quad (2.21)$$

Letting  $\varepsilon, \delta \rightarrow 0$  in (2.19) we see that

$$U^n(t) = \int_{S^1} (u_1(x, 0) - u_2(x, 0)) dx + A^n(t) + M^n(t), \quad (2.22)$$

where  $A^n(t) = \lim_{\varepsilon, \delta \rightarrow 0} \int_0^t C^{n, \varepsilon, \delta}(s) ds$  is a continuous, adapted non-increasing process. Now, sending  $n$  to  $\infty$  we obtain

$$U(t) = \int_{S^1} (u_1(x, 0) - u_2(x, 0)) dx + A(t) + M(t), \quad (2.23)$$

where

$$U(t) = \int_{S^1} (u(x, t) - v(x, t)) dx,$$

$$\begin{aligned} M(t) &= \int_0^t \int_{S^1} \sigma(u(x, s)) W_1(dx, ds) \\ & \quad - \int_0^t \int_{S^1} \sigma(v(x, s)) (1 - |u - v| \wedge 1)^{\frac{1}{2}} \dot{W}_1(dx, ds) \\ & \quad - \int_0^t \int_{S^1} \sigma(v(x, s)) (|u - v| \wedge 1)^{\frac{1}{2}} \dot{W}_2(dx, ds), \end{aligned}$$

and  $A(t) = \lim_{n \rightarrow \infty} A^n(t)$  a continuous, adapted non-increasing process. The existence of the limits of  $A^n$  follows from the existence of the limit of  $U^n$  and

$M^n$ . Now we can modify the proof in [3] to obtain the successful coupling of  $u$  and  $v$ . In view of the assumption on  $\sigma$  and the boundedness of the walls  $h^1, h^2$ , it is easy to verify that

$$\frac{d \langle M \rangle (t)}{dt} \geq C_0 U(t) \quad (2.24)$$

for some positive constant  $C_0$ . Thus, there exists a non-negative adapted process  $V(t)$  such that

$$\frac{d \langle M \rangle (t)}{dt} = U(t)V(t), \quad V(t) \geq C_0.$$

Let

$$\begin{aligned} \phi(t) &= \int_0^t V(s) ds, \\ X(t) &= U(\phi^{-1}(t)). \end{aligned} \quad (2.25)$$

Then the time-changed process  $X$  satisfies the following equation

$$X(t) = U(0) + \tilde{A}(t) + \int_0^t X^{\frac{1}{2}}(s) dB(s), \quad (2.26)$$

where  $B$  is a Brownian motion and  $\tilde{A}$  is an adapted non-increasing process. Let  $Y(t) = 2X^{\frac{1}{2}}(t)$ . Applying Ito's formula (before  $Y$  hits 0) we obtain

$$Y(t) = Y(0) + 2 \int_0^t \frac{1}{Y(s)} d\tilde{A}(s) - \frac{1}{2} \int_0^t \frac{1}{Y(s)} ds + B(t). \quad (2.27)$$

As  $\tilde{A}$  is non-increasing, it follows that

$$0 \leq Y(t) \leq Y(0) + B(t). \quad (2.28)$$

The property of one dimensional Brownian motion implies that  $Y$  hits 0 with probability 1. Hence

$$\lim_{t \rightarrow \infty} P\left(\sup_{x \in S^1} |u(x, t) - v(x, t)| \neq 0\right) = 0.$$

Next let us consider the general case, i.e. we do not assume  $u^1(x, 0) \geq u^2(x, 0)$ ,  $x \in S^1$ . Consider a solution  $v, u^1, u^2$  of the following SPDEs with

two reflecting walls

$$\begin{aligned}
\frac{\partial v(x, t)}{\partial t} &= \frac{\partial^2 v(x, t)}{\partial x^2} + f(v(x, t)) + \sigma(v(x, t)) \dot{W}_1(x, t) \\
&\quad + \eta_v(x, t) - \xi_v(x, t), \\
\frac{\partial u^i(x, t)}{\partial t} &= \frac{\partial^2 u^i(x, t)}{\partial x^2} + f(u^i(x, t)) + \eta_{u^i}(x, t) - \xi_{u^i}(x, t) \\
&\quad + \sigma(u^i(x, t)) [(1 - |v - u^i| \wedge 1)^{\frac{1}{2}} \dot{W}_1(x, t) + (|v - u^i| \wedge 1)^{\frac{1}{2}} \dot{W}_2(x, t)], \\
v(x, 0) &= \max_{i=1,2} \{u^i(x, 0)\}.
\end{aligned}$$

By following the arguments in the first part, we have

$$\lim_{t \rightarrow \infty} P \left( \sup_{x \in S^1} |v(x, t) - u^i(x, t)| \neq 0 \right) = 0, \quad i = 1, 2.$$

The inequality

$$0 \leq \sup_{x \in S^1} |u^1(x, t) - u^2(x, t)| \leq \sum_{i=1}^2 \left( \sup_{x \in S^1} |v(x, t) - u^i(x, t)| \right)$$

implies

$$\lim_{t \rightarrow \infty} P \left( \sup_{x \in S^1} |u^1(x, t) - u^2(x, t)| \neq 0 \right) = 0.$$

□

### 3 Strong Feller property

In this section, we consider the strong Feller property of the solution of equation (1.1). Let  $H = L^2(S^1)$ . If  $\varphi \in B_b(H)$  (the Banach space of all real bounded Borel functions, endowed with the sup norm), we define, for  $x \in S^1$ ,  $0 \leq t \leq T$  and  $g \in H$ ,

$$P_t \varphi(g) = \mathbb{E} \varphi(u(x, t, g)).$$

**Defintion 3.1.** *The family  $\{P_t\}$  is called strong Feller if for arbitrary  $\varphi \in B_b(H)$ , the function  $P_t \varphi(\cdot)$  is continuous for all  $t > 0$ .*

**Theorem 3.1.** *Under the hypotheses (H1)-(H2), (F1)-(F3) and that  $p_1 \leq |\sigma(\cdot)| \leq p_2$  for some constants  $p_1, p_2 > 0$ , then for any  $T > 0$  there exists a constant  $C'_T$  such that for all  $\varphi \in B_b(H)$  and  $t \in (0, T]$ ,*

$$|P_t \varphi(u_0^1) - P_t \varphi(u_0^2)| \leq \frac{C'_T}{\sqrt{t}} \|\varphi\|_\infty |u_0^1 - u_0^2|_H, \quad (3.1)$$

for  $u_0^1, u_0^2 \in H$  with  $h^1(x) \leq u_0^1(x)$ ,  $u_0^2(x) \leq h^2(x)$ , where  $\|\varphi\|_\infty = \sup_{u_0} |\varphi(u_0)|$ .

In particular,  $P_t$ ,  $t > 0$ , is strong Feller.

**Proof.** Choose a non-negative function  $\phi \in C_0^\infty(R)$  with  $\int_R \phi(x) = 1$  and denote

$$f_n(\zeta) = n \int_R \phi(n(\zeta - y)) f(y) dy,$$

$$\sigma_n(\zeta) = n \int_R \phi(n(\zeta - y)) \sigma(y) dy,$$

$$k_n(\zeta, x) = n \int_R \phi(n(\zeta - y)) (y - h^1(x))^- dy,$$

$$l_n(\zeta, x) = n \int_R \phi(n(\zeta - y)) (y - h^2(x))^+ dy.$$

So  $f_n$ ,  $\sigma_n$ ,  $k_n$ ,  $l_n$  are smooth w.r.t.  $\zeta$ . Let

$$\begin{aligned} u_n^{\varepsilon, \delta}(x, t, u_0) &= \int_{S^1} G_t(x, y) u_0(y) dy + \int_0^t \int_{S^1} G_{t-s}(x, y) f_n(u_n^{\varepsilon, \delta}(y, s, u_0)) dy ds \\ &\quad + \int_0^t \int_{S^1} G_{t-s}(x, y) \sigma_n(u_n^{\varepsilon, \delta}(y, s, u_0)) W(dy, ds) \\ &\quad + \frac{1}{\delta} \int_0^t \int_{S^1} G_{t-s}(x, y) k_n(u_n^{\varepsilon, \delta}(y, s, u_0), y) dy ds \\ &\quad - \frac{1}{\varepsilon} \int_0^t \int_{S^1} G_{t-s}(x, y) l_n(u_n^{\varepsilon, \delta}(y, s, u_0), y) dy ds. \end{aligned}$$

Since  $f_n(\zeta) \rightarrow f(\zeta)$ ,  $\sigma_n(\zeta) \rightarrow \sigma(\zeta)$ ,  $k_n(\zeta, x) \rightarrow (\zeta - h^1(x))^-$  and  $l_n(\zeta, x) \rightarrow (\zeta - h^2(x))^+$  as  $n \rightarrow \infty$ , we can show that for any fixed  $\varepsilon$ ,  $\delta$  and  $p \geq 1$ ,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{E}(|u_n^{\varepsilon, \delta}(t, \cdot, u_0) - u^{\varepsilon, \delta}(t, \cdot, u_0)|_H^p) = 0.$$

By Lemma 7.1.5 in [2] and Proposition 2.1, it is enough to prove that there exists a constant  $C'_T$ , independent of  $\varepsilon$ ,  $\delta$  and  $n$ , such that

$$|P_t^{n, \varepsilon, \delta} \varphi(u_0^1) - P_t^{n, \varepsilon, \delta} \varphi(u_0^2)| \leq \frac{C'_T}{\sqrt{t}} \|\varphi\|_\infty |u_0^1 - u_0^2|_H, \quad (3.2)$$

where  $P_t^{n, \varepsilon, \delta} \varphi(u_0) := \mathbb{E}(\varphi(u_n^{\varepsilon, \delta}(\cdot, \cdot, u_0)))$  and  $u_0^1, u_0^2 \in H$ .

From Theorem 5.4.1 in [2],  $u_n^{\varepsilon,\delta}(\cdot, \cdot, u_0)$  is continuously differentiable w.r.t.  $u_0$ . Denote by  $X_n^{\varepsilon,\delta}(x, t) := (Du_n^{\varepsilon,\delta}(\cdot, \cdot, u_0)(\bar{u}_0))(x, t)$  the directional derivative of  $u_n^{\varepsilon,\delta}(\cdot, \cdot, u_0)$  at  $u_0$  in the direction of  $\bar{u}_0$  and it satisfies the mild form of a SPDE

$$\begin{aligned} X_n^{\varepsilon,\delta}(x, t) &= \int_{S^1} G_t(x, y) \bar{u}_0(y) dy + \int_0^t \int_{S^1} G_{t-s}(x, y) f'_n(u_n^{\varepsilon,\delta}(y, s, u_0)) X_n^{\varepsilon,\delta}(y, s) dy ds \\ &\quad + \int_0^t \int_{S^1} G_{t-s}(x, y) \sigma'_n(u_n^{\varepsilon,\delta}(y, s, u_0)) X_n^{\varepsilon,\delta}(y, s) W(dy, ds) \\ &\quad + \frac{1}{\delta} \int_0^t \int_{S^1} G_{t-s}(x, y) \frac{\partial}{\partial \zeta} k_n(u_n^{\varepsilon,\delta}(y, s, u_0), y) X_n^{\varepsilon,\delta}(y, s) dy ds \\ &\quad - \frac{1}{\varepsilon} \int_0^t \int_{S^1} G_{t-s}(x, y) \frac{\partial}{\partial \zeta} l_n(u_n^{\varepsilon,\delta}(y, s, u_0), y) X_n^{\varepsilon,\delta}(y, s) dy ds. \end{aligned}$$

Since  $\frac{\partial}{\partial \zeta} k_n(u_n^{\varepsilon,\delta}(y, s, u_0), y) \leq 0$ ,  $\frac{\partial}{\partial \zeta} l_n(u_n^{\varepsilon,\delta}(y, s, u_0), y) \geq 0$ , we use the similar arguments as that in [14] and to get

$$\sup_{\varepsilon, \delta \geq 0, t \in [0, T]} \mathbb{E} \left( \int_{S^1} (X_n^{\varepsilon,\delta}(y, t))^2 dy \right) \leq C |\bar{u}_0|_H^2,$$

where  $C$  is a constant. By Elworthy-Li formula (Lemma 7.1.3 in [2]), we obtain

$$| \langle DP_t \varphi(u_0), \bar{u}_0 \rangle |^2 \leq \frac{C}{p_1^2(t)} \|\varphi\|_\infty^2 |\bar{u}_0|_H^2.$$

This implies inequality (3.2) which completes the proof.  $\square$

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